Regions of multistability in some low-dimensional logistic models with excitation type coupling

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Abstract— A naive model of many networked logistic maps with an excitation type coupling [Neural Networks, vol. 20, 102–108 (2007)], which is an extension of other low dimensional models, has been recently proposed to mimic the waking-sleeping bistability found in brain systems. Although the dynamics of large and complex aggregates of elementary components can not be understood nor extrapolated from the properties of a few components, some patterns of behavior could be conserved independently of the topology and of the number of coupled units. Following this insight, we have collected several of those systems where a few logistic maps are coupled under a similar mutual excitation scheme. The regions of bi- and multistability of these systems are sketched and reported.

I. INTRODUCTION

The brain is a natural networked system [1]. The understanding of this complex system is one of the most fascinating scientific tasks today, concretely how this set of millions of neurons can interact among them to give rise to the collective phenomenon of human thinking [2], or, in a simpler and more realistic approach, what neural features can make possible, for example, the birdsongs [3].

ANIMAL	BRAIN WEIGHT (gr.)	SLEEP TIME (hours)
Cat	30	13
Dog	70	11
Human (-)	350	16
Human (+)	1300	8
Horse	530	2
Elephant	4500	4

Fig. 1. Brain size and mean sleeping time for different animals. (Humans(-) represent new born humans and Humans(+) represent middle age humans).

What is clear is that many of the brain functions are not dependent neither of its architecture nor of its size. Take, for instance, the universal property in mammals and birds of the sleep-wake cycle [4], [5], a regular daily behavior that is closely synchronized with the cycle of sun light. It is observed that, in general, large animals tend to sleep less than small animals (some sleeping time data are shown in Fig. 1). Hence, the bistable sleep-wake behavior is one of the emergent attributes that does not depend on the precise architecture of the brain nor on its size [6], [7]. If we represent the brain as a complex network this property would mean that this possible bistability where large groups of neurons can show some kind of synchronization should not depend on the topology (structure) nor on the number of nodes (size) of the network (Fig. 2).
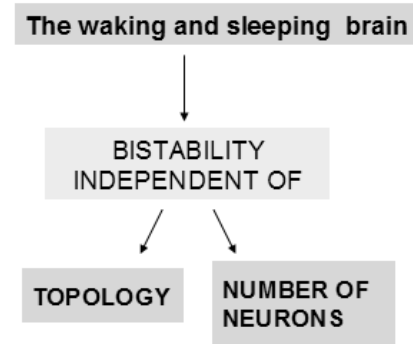

Fig. 2. The waking-sleeping brain.

Then it seems that it is essential the type of local dynamics and coupling among nodes that must be implemented in order to get bistability as the possible dynamical panorama of a complex network. In the next section, we give some possible (excitation/inhibition) strategies for the coupling and the local (logistic) dynamics which should be implemented in a few or many units network in order to find bistable behavior between two complete synchronized states [7]. In Section 3, some discrete logistic models in two and three dimensions, showing bi- and multistability under the excitation scheme, are presented. Finally, the last section contains our conclusions.

II. GENERAL MODEL

Our approach consider the so called *functional unit*, i.e. a neuron or group of neurons, as a discrete nonlinear oscillator [8] with two possible states: active (meaning one type of activity) or not (meaning other type of activity). Hence, in this naive vision of the brain as a networked system, if x_n^i, with $0 < x_n^i < 1$, represents a measurement of the ith functional unit activity at time n, it can be reasonable to take the most elemental local nonlinearity, for instance, a logistic evolution

[9], [10], which presents a quadratic term, as a first toy-model for the local neuronal activity (Fig. 3):

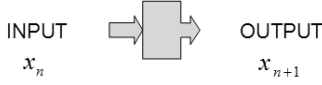$$x_{n+1}^i = \bar{p}_i \, x_n^i (1 - x_n^i). \tag{1}$$

Fig. 3.   Discrete nonlinear model for the local evolution of a functional unit.

It presents only one stable state for each $\bar{p}_i$. Then, there is no bistability in the basic component of our models. For $\bar{p}_i < 1$, the dynamics dissipates to zero, $x_n^i = 0$, then it can represent the functional unit with no activity. For $1 < \bar{p}_i < 4$, the dynamics is non null and it would represent an active functional unit.

We can suppose that this local parameter $\bar{p}_i$ is controlled by the signals of neighbor units, simulating in some way the effect of the synapses among neurons. Excitatory and inhibitory synaptic couplings have been shown to be determinant on the synchronization of neuronal firing. For instance, facilitatory connections are important to explain the neural mechanisms that make possible the object representation by synchronization in the visual cortex [11]. While it seems clear that excitatory coupling can lead to synchronization, frequently inhibition rather than excitation synchronizes firing [12]. The importance of these two kinds of coupling mechanisms has also been studied for other types of neurons, v.g., motor neurons [13].

If a neuron unit simultaneously processes a plurality of binary input signals, we can think that this local information processing is reflected by the parameter $\bar{p}_i$. The functional dependence of this local coupling on the neighbor states is essential in order to get a good brain-like behavior (i.e., as far as the bistability of the sleep-wake cycle is concerned) of the network. As a first approach, we can take $\bar{p}_i$ as a linear function depending on the actual mean value, X_n^i, of the neighboring signal activity and expanding the interval $(1, 4)$ in the form:

$$\bar{p}_i = p_i (3X_n^i + 1), \quad (excitation\ coupling) \tag{2}$$
$$or$$
$$\bar{p}_i = p_i (-3X_n^i + 4), \quad (inhibition\ coupling) \tag{3}$$

with

$$X_n^i = \frac{1}{N_i} \sum_{j=1}^{N_i} x_n^j. \tag{4}$$

N_i is the number of neighbors of the ith functional unit, and p_i, which gives us an idea of the interaction of the functional unit with its first-neighbor functional units, is the control parameter. This parameter runs in the range $0 < p_i < p_{max}$, where $p_{max} \succeq 1$. When $p_i = p$ for all i, the dynamical behavior of these networks with the excitation type coupling [6] presents an attractive global null configuration that has

been identified as the *turned off* state of the network. Also they show a completely synchronized non-null stable configuration that represents the *turned on* state of the network. Moreover, a robust bistability between these two perfect synchronized states is found in that particular model (see [6] and [14] for more details). For different models with a few coupled functional units we sketch in the next subsections the regions where they present a multistable behavior.

III. MODELS WITH EXCITATION COUPLING

A. Model of two functional units

The case of two interconnected (x_n, y_n) functional units [15] under the *(excitation, excitation)* scheme is given by the coupled equations:

$$x_{n+1} = p\,(3y_n + 1)x_n(1 - x_n), \tag{5}$$
$$y_{n+1} = p\,(3x_n + 1)y_n(1 - y_n). \tag{6}$$

The regions of the parameter space where we can find bistability was presented in [7].

B. Models of three functional units

1) Model with local mutual excitation: The case of three alternatively interconnected (x_n, y_n, z_n) functional units [16] under a mutual excitation scheme is given by the coupled equations:

$$x_{n+1} = p\,(3y_n + 1)x_n(1 - x_n), \tag{7}$$
$$y_{n+1} = p\,(3z_n + 1)y_n(1 - y_n), \tag{8}$$
$$z_{n+1} = p\,(3x_n + 1)z_n(1 - z_n). \tag{9}$$

The regions of the parameter space where bistability can be found are sketched in [7].

2) Model with global mutual excitation: The case of three globally interconnected (x_n, y_n, z_n) functional units [16] under a mutual excitation scheme is given by the coupled equations:

$$x_{n+1} = p\,(x_n + y_n + z_n + 1)x_n(1 - x_n), \tag{10}$$
$$y_{n+1} = p\,(x_n + y_n + z_n + 1)y_n(1 - y_n), \tag{11}$$
$$z_{n+1} = p\,(x_n + y_n + z_n + 1)z_n(1 - z_n). \tag{12}$$

For the whole range of the parameter, $0 < p < 1.17$, bistability is present in this system [7].

3) Model with partial mutual excitation: The new case of three partially interconnected (x_n, y_n, z_n) functional units under a mutual excitation scheme is represented in Fig. 4.
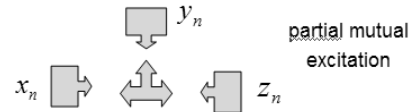

Fig. 4.   Three partially coupled functional units under the excitation scheme.

The dynamics of the system is given by the coupled equations:

$$x_{n+1} = p\left(3(y_n + z_n)/2 + 1\right)x_n(1 - x_n), \quad (13)$$

$$y_{n+1} = p\left(3(x_n + z_n)/2 + 1\right)y_n(1 - y_n), \quad (14)$$

$$z_{n+1} = p\left(3(x_n + y_n)/2 + 1\right)z_n(1 - z_n). \quad (15)$$

The rough inspection of this system puts in evidence the existence of different regions of multistability in the parameter space. These are:

- For $0.93 < p < 1.04$, there are coexistence among three cycles of period-2.
- For $1.04 < p < 1.06$, the three cycles bifurcate giving rise to three order-2 ICC (Fig. 5).
- For $1.06 < p < 1.08$, the system can present three mode-locked periodic orbits or three chaotic cyclic attractors, each one with period multiple of 6, or three chaotic cyclic attractors of order 2 (Fig. 6).
- For $p > 1.08$, the chaotic cyclic attractors collapse in an unique chaotic attractor (Fig. 7).

Also, other bistable and multistable situations can be found for some particular values of the parameter p in the former intervals.
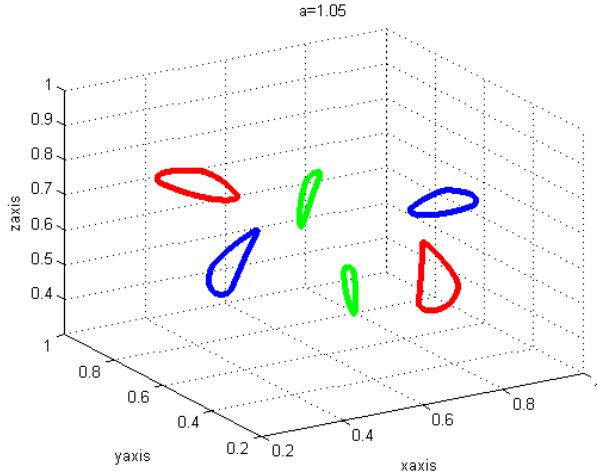

Fig. 6. Multistability of three chaotic cyclic attractors of order 2 for $p = a = 1.078$.

Fig. 7. An unique chaotic attractor for $p = a = 1.09$.

Fig. 5. Multistability in 3 functional units with partial excitation type coupling. The system presents three order-2 ICC for $p = a = 1.05$.

IV. CONCLUSIONS

Different neural systems can exhibit similar dynamical properties despite having different architectures, different sizes or different complexity [17]. To be able of reproducing, even qualitatively, some common features observed in those systems could be considered as an effective advance. Thus, specifically, if the sleep-wake cycle is interpreted as a bistability in the global behavior of a neural system, it might be of some interest to dispose of a model that reproduces this type of phenomenology. In this presentation, different coupling schemes for networks with local logistic dynamics are proposed. It i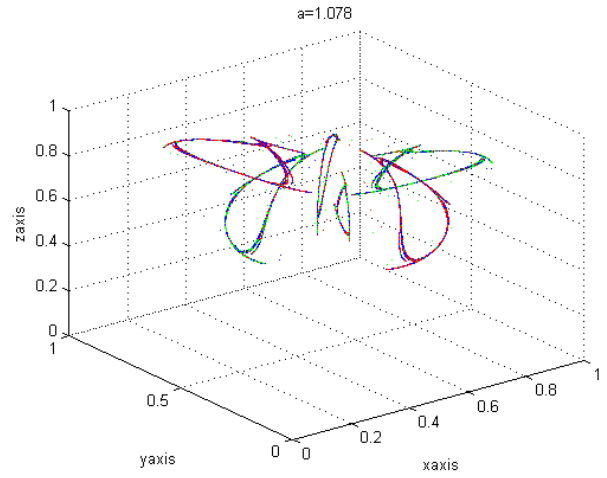s observed that this kind of couplings generates a global bistability between two different dynamical states. This property seems to be topology and size independent. This is a direct consequence of the local mean-field multiplicative coupling among the first-neighbors. Following this insight, different low-dimensional systems with logistic components coupled under an excitation scheme have been collected, and the regions where the dynamics shows (multi-) bistability have been identified.

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